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On the governing equations in relaxing media models and self-similar quasiperiodic solutions

V A Danylenko, V V Sorokina and V A Vladimirov

Subbotin Geophysical Institute of the Ukrainian Academy of Sciences Division of Geodynamics of Explosion, Lenin Street 63 B, 252054 Kiev, Ukraine

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Abstract. Dynamical equations of state for multicomponent relaxing media have been proposed. Using special ansatz, systems of hydrodynamical equations, closed by the equations of state proposed in this work, have been reduced to ODE systems, which were shown to possess, under certain conditions, sets of quasiperiodic solutions.

1. Introduction

Analysis of various numerical experiments with different continuum mechanics equations employed to describe relaxing high-rate processes in active media [1, 2] enables one to determine that the qualitative behaviour of their solutions is strongly influenced by the properties of the governing equations.

Multicomponent condensed media state equations are constructed, as a rule, by means of pure mechanical models, being postulated without rigorous thermodynamical substantiation. Because an equation of state describing properly condensed media in high-rate processes for a wide range of thermodynamical parameters does not exist, it seems reasonable to treat phenomena of relaxation as unknown chemical reactions. Using such an approach it is possible to obtain restrictions on the governing equations by means of non-equilibrium thermodynamics methods.

Restrictions on the state and kinetic equations could also be obtained from symmetry principles. On the other hand, when continuum mechanics equations admit a subsequently large symmetry group, it is possible to reduce the number of independent variables and, in some special cases, to pass from partial differential equations (PDE) to a system of ordinary differential equations (ODE). This finally gives the opportunity, taking advantage of qualitative theory methods, to connect properties of self-similar solutions of the initial system with features of the governing equations.

In the second part of this work, a system of PDE aimed at describing relaxation processes in fluids is analysed. From the general system, under certain conditions, a dynamical equation of state, included as a particular case of the generalized Maxwell equation and Lyakhov's equation [3], have been obtained.

In the third section, based on the group classification of hydrodynamical equations for active media performed in [1, 5], ansatz enabling one to pass from PDE to ODE of low dimensions have been proposed. The dynamical systems obtained were investigated by means of qualitative theory methods. This method shows that the initial system

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possesses families of self-similar quasiperiodic solutions for a wide class of governing equations.

2. State and kinetic equations in relaxing media models

Analysis of various experimental studies of wave propagation in multicomponent active media for constant temperature, pressure and number of molal particles (components) [3] enables one to conclude that some internal state variations (e.g. of volume) are possible. The molal number is the ratio of matter mass to molal mass (molecular weight). Phenomena arising then as an after-action or relaxation process might be formally described as unknown chemical reactions with the corresponding degree of reaction completeness. The reaction completeness degree is an internal state parameter added to the rest of the parameters in such a way that the internal state is defined macroscopically, though the relaxation mechanism associated with the internal parameters remains unknown. In this case, the non-equilibrium state of the system must be characterized not only by external parameters (volume, temperature), but also by an internal parameter [2, 5], which we denote as λ.

When treating strains hydrostatically it is possible to write down a closed system describing the processes in the relaxing media as follows:

$$\rho(u^i_t + u^j u^i_{x_j}) + p_{x_i} = \mathscr{F}^i \tag{1}$$

$$\rho_t + u^i \rho_{x_i} + \rho u^i_{x_i} = 0 \tag{2}$$

$$p_t + u^i p_{x_i} + M u^i x_i = N \tag{3}$$

$$\lambda_t + u^i \lambda_{x_i} = Q \tag{4}$$

where ρ is density, p is pressure, u^i are speed components $(i = 1, \ldots, n)$ λ is the intrinsic variable, $\mathscr{F}^i$ are the external force components. For adiabatic processes M and N are connected with intrinsic energy $\mathscr{E}(\rho, p, \lambda)$ and kinetic function $Q(\rho, p, \lambda)$ by the following relations:

$$M = (p - \rho^2 \mathscr{E}_\rho)/(\rho \mathscr{E}_p), \qquad N = -Q\mathscr{E}_\lambda/\mathscr{E}_p. \tag{5}$$

Let the functions $Q(\rho, T, \lambda)$ and $\rho^{-1} = V(p, T, \lambda)$ be expanded near the equilibrium $Q(\rho_0, T_0, \lambda_0) = 0$ into the power series

$$\begin{aligned} Q &= \left(\frac{\partial Q}{\partial T}\right)_{p,\lambda}(T - T_0) + \left(\frac{\partial Q}{\partial p}\right)_{T,\lambda}(p - p_0) + \left(\frac{\partial Q}{\partial \lambda}\right)_{T,p}(\lambda - \lambda_0) + \ldots \\ V - V_0 &= \left(\frac{\partial V}{\partial T}\right)_{p,\lambda}(T - T_0) + \left(\frac{\partial V}{\partial p}\right)_{T,\lambda}(p - p_0) + \left(\frac{\partial V}{\partial \lambda}\right)_{T,P}(\lambda - \lambda_0) + \ldots \end{aligned} \tag{6}$$

where $V_0 = V(P_0, T_0, \lambda_0)$ is the specific volume in the equilibrium state.

On excluding the terms $\lambda - \lambda_0$ and $\mathrm{d}\lambda/\mathrm{d}t$ from (4) and (6), taking derivative of (6) with respect to time and making use of the first law of thermodynamics we obtain the following relation:

$$\begin{aligned} p - p_0 = {} & \rho_0 C^2_{s\infty}/\gamma_\infty[\sigma(S)(V/V_0)^{-\gamma_\infty} - 1] - \tau_{TV}\,\mathrm{d}p/\mathrm{d}t - \tau_{TP}/(\chi_T V_0)\,\mathrm{d}V/\mathrm{d}t \\ & - \Gamma_V C_V \tau_{pV}/V_0\,\mathrm{d}T/\mathrm{d}t \end{aligned} \tag{7}$$

where S is entropy, $C_{s\infty}$ is the adiabatic sound velocity in the state of dynamical (frozen) equilibrium ($\omega \Rightarrow \infty$, $\omega = t_0^{-1}$ is the characteristic frequency of the process), $\mathrm{d}/\mathrm{d}t = \partial/\partial t + u^i\,\partial/\partial x^i$

$$\Gamma_V = \frac{V}{\chi_T C_{V\infty} V_0}\left(\frac{\partial V}{\partial T}\right)_{p\lambda} \qquad \chi_T = \frac{V}{C_{T\infty}^2}$$

$C_{V\infty}$ is the thermal capacity for constant volume in a state of dynamical (frozen) equilibrium ($\omega \Rightarrow \infty$), $C_{T\infty}$ is the isothermal sound velocity when $\omega \Rightarrow \infty$, $\gamma_\infty = \Gamma_V + 1$, $\rho_0 = V_0^{-1}$

$$\sigma(S) = (\gamma_\infty - 1) C_{V\infty} V_0^{1-\gamma_\infty} C_{T\infty}^{-2} \exp\left(\frac{S_0 - S}{C_{V\infty}}\right).$$

According to [6] we use designations $(\partial Q/\partial\lambda)_{AB} = \tau_{AB}^{-1}$. Quantities τ_{PV}, τ_{TV}, τ_{PV} are called relaxation times.

We investigate equation (7) for different τ_{TV}/t_0, τ_{PV}/t_0, τ_{TP}/t_0 ratios.

For $\tau_{PV}/t_0 \ll 1$, $S = \text{constant}$, $\sigma(S) = 1$, the equation formally identical to that for multicomponent media [3] follows from equation (7):

$$p - p_0 = \frac{\rho_0 C_{S\infty}^2}{\gamma_\infty}\left\{\left(\frac{V}{V_0}\right)^{-\gamma_\infty} - 1 + \left(\frac{V}{V_0}\right)^{-1}\right\} - \frac{\eta_V}{V_0}\frac{\mathrm{d}V}{\mathrm{d}t} - \tau_{TV}\frac{\mathrm{d}p}{\mathrm{d}t}. \tag{8}$$

Here $\eta_V = (\chi_T/\tau_{TP})^{-1}$ is volume viscosity index.

The term $(V/V_0)^{-1}$ may be neglected for condensed media.

For $\tau_{PV}/t_0 \ll 1$, $\tau_{TP}/t_0 \ll 1$ one can obtain from (7):

$$p - p_0 = \frac{\rho_0 C_{S\infty}^2}{\gamma_\infty}\left(\sigma(S)\left(\frac{V}{V_0}\right)^{-\gamma_\infty} - 1 + \left(\frac{V}{V_0}\right)^{-1}\right) - \tau_{TV}\frac{\mathrm{d}p}{\mathrm{d}t} \tag{9}$$

which is a nonlinear generalization of the Maxwell equation.

For $\tau_{TV}/t_0 \ll 1$, $\tau_{pv}/t_0 \ll 1$, $\sigma(S) = 1$ we obtain from (7) the following equation:

$$p - p_0 = \frac{\rho_0 C_{S\infty}^2}{\gamma_\infty}\left(\left(\frac{V}{V_0}\right)^{-\gamma_\infty} - 1 + \left(\frac{V}{V_0}\right)^{-1}\right) - \frac{\eta_V}{V_0}\frac{\mathrm{d}V}{\mathrm{d}t}. \tag{10}$$

Equation (10) was postulated by Lyakhov when deriving the governing equation for relaxing components [3].

In the limiting case when $\tau_{TV}/t_0 \Rightarrow 0$, $\tau_{PV}/t_0 \Rightarrow 0$, $\tau_{TP}/t_0 \Rightarrow 0$, equation (7) becomes identical to the Tait equation of state:

$$p - p_0 = \frac{\rho_0 C_{S0}^2}{\gamma_0}\left(\sigma(S)\left(\frac{V}{V_0}\right)^{-\gamma_0} - 1\right) \tag{11}$$

used in the absence of relaxing processes for a complete equilibrium state.

The applicability region of equations (8)–(10) describing viscous and viscoelastic media can be obtained by means of (7). It is known, from the general theory of non-equilibria processes, that for condition $\tau_i/t_0 \ll 1$ the relaxation process may be described as a 'viscous' process, while the remaining $(n-1)$ processes must be considered as relaxation processes. Only if all the relaxing times τ_i for the frequency band $\omega = t_0^{-1}$ (frequencies used in the measurements) satisfy the condition $\tau_i\omega \ll 1$, can we say that the relaxation process is of visco-elastic origin.

3. On the self-similar quasiperiodic solutions of hydrodynamical systems for active media

Throughout the remainder of this work we concern ourselves with self-similar solutions investigations [7, 8].

Let us consider the hydrodynamical system closed by (8):

$$
\begin{aligned}
& u_t^i + u^j u_{x_j}^i + \rho^{-1}\tilde{p}_{x_i} = \mathscr{F}^i \\
& \rho_t + u^i \rho_{x_i} + \rho u_{x_i}^i = 0 \\
& \tilde{p}_t + u^i \tilde{p}_{x_i} + \frac{L}{\rho} u_{x_i}^i = \kappa \rho^\gamma - \tilde{p}.
\end{aligned} \tag{12}
$$

Here the transition to dimensionless variables is made: $t \Rightarrow t/\tau_{\mathrm{TV}}$, $x \Rightarrow x(\rho_0/p_0)^{1/2}/\tau_{\mathrm{TV}}$, $u \Rightarrow u(\rho_0/p_0)^{1/2}$, $\rho \Rightarrow \rho/\rho_0$, $p \Rightarrow \tilde{p} = p_0^{-1}[p + (\rho_0 C_{\mathrm{S0}}^2)/\gamma - p_0]$, $\kappa = \rho_0 C_{\mathrm{S0}}^2/(\gamma p_0)$, $L = \eta_{\mathrm{V}} \rho_0^2/\tau_{\mathrm{TV}}$. For simplicity, the tilde over the p variable will be omitted.

By straightforward calculations one can determine that system (12) is invariant under the Galilei algebra $AG(n)$ spanned by the following operators:

$$
P_0 = \frac{\partial}{\partial t} \qquad P_i = \frac{\partial}{\partial x_i} \qquad J_{ab} = x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a} + u_a \frac{\partial}{u_b} - u_b \frac{\partial}{\partial u_a} \tag{13}
$$

$$
G_a = t\frac{\partial}{\partial x_a} + \frac{\partial}{\partial u_a} \qquad i, a, b = 1, \ldots, n. \tag{14}
$$

For some special cases the symmetry of (12) is wider than $AG(n)$. If $L = 0$, $\gamma = 1$, and $\mathscr{F} = \gamma\rho$, this system admits an extra one-parameter group generated by the operator $D = \rho\partial/\partial\rho + p\partial/\partial p$. This makes it possible to choose some new variables enabling one to pass, in the case of one spatial variable from (12), to a system of two ODE plus one quadrature. Using the ansatz

$$
\begin{aligned}
& u = \beta + U(w) \qquad w = x - \beta t \\
& \rho = \exp[\xi t + S(w)] \qquad p = Z(w)\exp[\xi t + S(w)]
\end{aligned} \tag{15}
$$

derived from the symmetry of (12) with respect to the group generated by $\hat{X} = P_0 + \beta P_1 + \xi D$, we obtain an ODE system which does not contain the S variable in explicit form. Functions U and Z satisfy the equations

$$
\begin{aligned}
& \frac{\mathrm{d}U}{\mathrm{d}\tau} = U(\kappa - \sigma Z - \gamma U) \equiv UF \\
& \frac{\mathrm{d}Z}{\mathrm{d}\tau} = ZF + U^2(Z - \kappa)
\end{aligned} \tag{16}
$$

Here $\mathrm{d}/\mathrm{d}\tau = -U^3\, d/dw$, $\sigma = 1 + \xi$.

The system (16) has the following critical points (CP): $A_1(U = O = Z)$, $A_2(U = 0, Z = \kappa)$, $B(U = -\xi\kappa/\gamma, Z = \kappa)$. For all values of the parameters, A_1 is an unstable nodal point. When σ approaches zero, A_2 goes to infinity. If $\sigma = 1$ then A_2 and B coincide. If $\xi < 0$ then A_2 is an unstable nodal point. If $\xi = 0$ then $B(A_2)$ is a saddle-nodal point. When $\xi > 0$, then A_2 is a saddle.

To analyse the behaviour of the critical point B with change of parameter values let us consider the linear part of the RHS of (16) in the vicinity of the point

$U_0 = -\xi\kappa/\gamma$, $Z_0 = \kappa$. Introducing the new variables $x = U - U_0$, $y = Z - Z_0$, inserting them into (16) and retaining only the linear part of the RHS we obtain

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \kappa\xi & \dfrac{\kappa\xi}{\gamma}\sigma \\ -\kappa\gamma & \kappa\left(\dfrac{\xi^2\kappa}{\gamma^2} - \sigma\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \equiv \hat{M} \begin{pmatrix} x \\ y \end{pmatrix} \tag{17}$$

Eigenvalues of the matrix M are given by

$$2\lambda_{1,2} = \kappa \left\{ \left(\frac{\xi}{\gamma}\right)^2 \kappa - 1 \pm \left\{ \left[\left(\frac{\xi}{\gamma}\right)^2 \kappa - 1 \right]^2 - 4\frac{\xi^3}{\gamma^2} \right\}^{1/2} \right\}. \tag{18}$$

It follows from (18) that if $\xi < 0$, then B is a saddle point. If $\xi > 0$ then B is stable node or focus.

Let us analyse the possibility of the limit cycle (LC) existence in the vicinity of B. When $\xi > 0$, there exists the manifold defined by the expression

$$\kappa - \left(\frac{\gamma}{\xi}\right)^2 = 0$$

where eigenvalues of $\hat{M}$ become purely imaginary. As a bifurcation parameter we take γ, assuming that $\xi = $constant and $\kappa = $constant. It is easy to very that, with these assumptions, Andronov–Hopf theorem conditions [9] are satisfied. So the parameter γ passing through the critical value $\gamma_{cr} = \pm\sqrt{\kappa\xi^2}$, shows the creation of a limit cycle.

Taking advantage of the rules for calculating the first Floquet index [9] we can state that the limit cycle is stable provided that $\gamma_{cr} < 0$.

The global phase portrait of system (16) for the case in question is presented in figure 1. Note that the directions of the trajectories in the RHS half-plane of figure 1

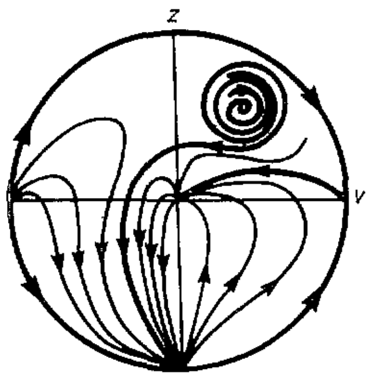

Figure 1. Global phase portrait of system (16).

are inverted. This is due to the 'time' signature change that occurs when passing to the old independent variable $\tau \Rightarrow w$.

So there exist values of the parameters for which the system (12) possess a family of quasiperiodic solutions.

In section 2 we introduced an intrinsic variable λ to describe hydrodynamical relaxing media and then obtained governing equations under the assumption that the processes described are not far from equilibrium. Now we investigate the properties of hydrodynamic systems for relaxing media without such a supposition.

To employ similarity theory methods for qualitative study of the system (1)–(4) solutions we look for special cases when state and kinetic functions are not rigidly fixed and, simultaneously, symmetry of the system is sufficiently large. Based on the group classification performed in [1, 4] it is possible to point out two cases when the kinetic function is quite general and, at the same time, symmetry of the system unables one, making group-theoretical reduction [7], to obtain two-dimensional systems of ODE.

In both cases the intrinsic energy is as follows:

$$\mathscr{E}=\frac{p}{\rho(\sigma-1)}-q\lambda \tag{19}$$

where $q>0$, $\sigma>1$. From (5) and (19) we obtain with ease that

$$M=\sigma p \qquad N=q(\sigma-1)\rho Q.$$

First we assume that $n=1$, $Q=g(\lambda)\varphi(p/\rho)t^{-1}$ and $\mathscr{F}=\rho\gamma t^{-1}$. With this assumption it is not difficult to see that the system (1)–(4) admits a one-parameter Lie group generated by the following operator:

$$\hat{Z}=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$$

Solving equation $\hat{Z}J=0$ we can build the following ansatz

$$\begin{aligned} &u=U(\omega)+\omega \qquad \omega=x/t \\ &\rho=\exp S(\omega) \qquad p=Z(\omega)\exp S(\omega) \qquad \lambda=L(\omega) \end{aligned} \tag{20}$$

Inserting (20) into (1)–(4), after some algebraic manipulation we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}U=U\{\Psi-\sigma Z-U(\gamma-U)\}\equiv U\Phi \tag{21}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}Z=(\sigma Z-U^2)[\Psi+(1-\sigma)Z]+Z(1-\sigma)\Phi \tag{22}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}L=(\sigma Z-U^2)Q \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}S=(U^2-\sigma Z)[1+\Phi] \tag{24}$$

where $\Psi=q(\sigma-1)Q$, $Q=g(L)\varphi(Z)$, $\mathrm{d}/\mathrm{d}\tau\equiv\Delta\,\mathrm{d}/\mathrm{d}\omega$, $\Delta=U(\sigma Z-U^2)$.

Let $g(L)\equiv 1$. In this case we can restrict ourselves to the analysis of the two-dimensional system (21), (22). One of the critical points of this system is given by the following equations:

$$U(\gamma-U)+Z=0 \tag{25}$$

$$\Psi=(\sigma-1)Z. \tag{26}$$

Assume that $Z_1>0$ is the solution of (26). Then another coordinate of CP could be obtained from (25):

$$U_1=\frac{\gamma+\varepsilon\sqrt{\gamma^2+4Z_1}}{2} \tag{27}$$

where $\varepsilon=\pm1$.

Let us determine whether or not the Andronov–Hopf theorem conditions are fulfilled. Function Ψ in the vicinity of the point Z_1 could be expressed in the following form:

$$\Psi=(\sigma-1)Z_1+\xi(Z-Z_1)+\eta(Z-Z_1)^2+\chi(Z-Z_1)^3+\dots. \tag{28}$$

Now passing to the variables $X=U-U_1$, $Y=Z-Z_1$ and linearizing (21) and (22) in the vicinity of the origin we obtain

$$\frac{d}{d\tau}\begin{bmatrix}X\\Y\end{bmatrix}=\begin{bmatrix}U_1(\gamma-2U_1), & U_1(\sigma-\xi)\\ Z_1(1-\sigma)(\gamma-2U_1), & (1-\sigma)U_1^2+U_1\gamma\xi\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}\equiv\hat{R}\begin{bmatrix}X\\Y\end{bmatrix}. \tag{29}$$

Eigenvalues of matrix $\hat{R}$ will be purely imaginary provided that sp $\hat{R}=0$ and det $\hat{R}>0$. From the first condition we obtain the following relation:

$$\gamma^2[\xi^2-\xi(\sigma-1)-\sigma]=Z_1(\sigma+1)^2. \tag{30}$$

The second condition will be satisfied if $\varepsilon=\gamma/|\gamma|$ (see formula (27)) and

$$\xi>\frac{\sigma^2+1}{\sigma-1}. \tag{31}$$

As a bifurcation parameter we choose γ. Denote a critical value (given by formula (30)) by γ_0. The real part of the operator $\hat{R}$ in the vicinity of the critical value γ_0 could be expressed as a function of the canonical parameter $\mu=\gamma-\gamma_0$:

$$\operatorname{Re}\lambda(\mu)=\mu\,\frac{\gamma_0(\xi-\sigma)(\sqrt{\gamma_0^2+4Z_1}-|\gamma_0|)^2}{4|\gamma_0|\sqrt{\gamma_0^2+4Z_1}}.$$

Now it is not difficult to see that the conditions of the Andronov–Hopf theorem are satisfied, and in the vicinity of the critical value $\mu=0$ creation of a limit cycle takes place.

To analyse the stability of LC we estimate both signs of Δ and the first Floquet index Re $C_1(\mu)$ when $\mu=0$. Taking advantage of (31) we easily obtain that

$$\operatorname{sgn}\Delta=-\operatorname{sgn}\gamma_0.$$

The estimation of $\mathrm{Re}_1\,C(0)$ sign is much more tedious than that of Δ so we reproduce here, without proof, the result obtained in [10]:

Proposition 1. When

$$\Psi=(\sigma-1)Z_1+\xi(Z-Z_1)+O(|Z-Z_1|^3) \tag{32}$$

then for any Z_1, γ and σ that satisfy (31) Re $C_1(0)>0$. In the general case, when the decomposition (28) has been performed, the following representation holds:

$$\text{Re } C_1(0)=F(|\gamma_0|, \sigma, Z_1). \tag{33}$$

Now we are able to formulate the following statement:

Theorem 1. If in the vicinity of a point $Z_1>0$, the kinetic function $\Psi(Z)$ has the decomposition (28) and parameters ξ, σ satisfy inequality (31) then there exists an open interval $I\subset R^1$ in the vicinity of the critical value

$$\gamma_0=\text{sgn}[\text{Re } C_1]\sqrt{Z_1}\,\frac{\sigma+1}{(\xi^2-\xi(\sigma-1)-\sigma)^{1/2}}$$

such that for $\gamma\in I$ the system (21), (22) possesses a family of stable quasiperiodic solutions.

Remark. Taking γ_0 with opposite sign we can obtain in the vicinity of this value a family of quasiperiodic unstable solutions.

The result obtained was verified by straightforward numerical calculations (figure 2). Function Ψ was taken in the simplest triangular form:

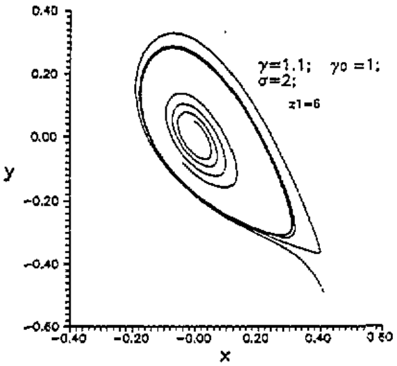

Figure 2. Trajectories of system (21), (22) in the vicinity of the critical point A obtained by straightforward numerical calculations by means of Runge-Kutta methods.

$$\Psi=\begin{cases}(\sigma-1)Z_1+\xi(Z-Z_1) & \text{if } Z_0<Z<Z_1+h\\ (\sigma-1)Z_1-\xi[Z-(Z_1+2h)] & \text{if } Z_1+h\leqslant Z\leqslant Z_2\\ 0 & \text{if } Z\in[Z_0, Z_2]\end{cases} \tag{34}$$

where $Z_0=Z_1\ (\xi-\sigma+1)/\xi$, $Z_2=2(Z_1+h)-Z_0\,h>0$.

Now let us consider the second case. Assume that $Q=Q(p/\rho)$ and $\mathscr{F}=\rho\gamma$, $\gamma=$ constant. It is not difficult to show that under these conditions the system (1)–(4) in the one-dimensional case ($n=1$) admits a Lie group generated by the operator

$$Z=P_0+\beta P_1+\alpha D. \tag{35}$$

Let us introduce the following ansatz based on the invariants of the operator (35):

$$u = U(\omega) \qquad \rho = \exp(\alpha t) R(\omega) \qquad p = \exp(\alpha t)\Pi(\omega), \qquad \lambda = L(\omega) \qquad \omega = x - \beta t. \tag{36}$$

Inserting (36) into (1)–(4) we obtain an ODE system which does not depend on the L variable explicitly, so we can reduce the problem to three-dimensional system analysis. One more equation could be dropped by taking advantage of the following ansatz:

$$U = W + \beta, \qquad R = \exp S(\omega) \qquad \Pi = Z \exp S(\omega) \tag{37}$$

making the S variable cyclic. Variables W and Z satisfy the following system:

$$\begin{aligned} \frac{dW}{d\tau} & W[\alpha Z + \gamma W - \psi(Z)] \\ \frac{dZ}{d\tau} &= Z(1-\sigma)[\gamma W + \alpha Z - \psi(Z)] + \psi(Z)(W^2 - \sigma Z) \end{aligned} \tag{38}$$

where $d/d\tau = \Delta d/d\omega$, $\Delta = W(W^2 - \sigma Z)$, $\psi(Z) = q(\sigma - 1)Q(Z)$.

Assume that $\psi(Z)$ intersects transversally the OZ axis at some point $Z_1 > 0$. Then the decomposition

$$\Psi = \xi(Z - Z_1) + \eta(Z - Z_1)^2 + \chi(Z - Z_1)^3 + \ldots. \tag{39}$$

with $\xi \neq 0$ holds.

The point C of phase plane with coordinates $W = -\alpha Z_1/\gamma$, $Z = Z_1$ is a critical point of (38). Passing to new variables $x = W + \alpha Z_1/\gamma$, $y = Z - Z_1$ and linearizing the system in the vicinity of the origin we obtain

$$\frac{d}{d\tau}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -\alpha Z_1, & \alpha Z_1(\zeta - \alpha)/\gamma \\ Z_1\gamma(1-\sigma), & Z_1\{[\alpha/\gamma)^2 Z_1 - 1]\zeta + (1-\sigma)\alpha\} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \equiv \hat{M} \begin{bmatrix} X \\ Y \end{bmatrix}$$

where

$$\zeta = d\psi/dZ|_{Z=Z_1}$$

From the condition sp $\hat{M} = 0$ we calculate α:

$$\alpha = \frac{\gamma^2\sigma + \varepsilon|\gamma|[(\sigma\gamma)^2 + 4Z_1\zeta]^{1/2}}{2Z_1\zeta} \qquad \varepsilon = \pm 1. \tag{40}$$

Analysis of the condition det $\hat{M} > 0$ gives $\varepsilon = +1$ and the following inequality:

$$Z_1\left[\frac{\xi}{\gamma}\right]^2 > \frac{\sigma^3}{(\sigma - 1)^2}. \tag{41}$$

As a bifurcation parameter we choose $\mu = \gamma - \gamma_0$ where γ_0 is a fixed value for which the inequality (41) is satisfied. Assuming that $\alpha = \alpha(\zeta, Z_1, \gamma_0)$, it is not difficult to show that the conditions of the Andronov–Hopf theorem [9] are satisfied, so μ, being in the vicinity of the critical value $\mu_{cr} = 0$, LC creation takes place.

The limit cycle is stable provided that Δ Re $C_1(\mu) < 0$ for $\mu = 0$. As in the previous case we are able to formulate the following statement [10]:

Proposition 2. Suppose that Ψ has the decomposition (39) and inequality (41) holds. Then the first Floquet index could be represented as follows:

$$\mathrm{Re}\, C_1(0)=\xi(\Phi^2+S\eta^2)+\chi K^2+\gamma H \tag{42}$$

where Φ, S, K and H are functions depending on σ, Z_1, $|\xi|$ and $|\gamma_0|$.

For Δ the following relations are true:

$$\begin{aligned}\mathrm{sgn}\,\Delta &= \mathrm{sgn}[W_1(W_1^2-\sigma Z_1)]=\mathrm{sgn}\left[\frac{\alpha}{\gamma}\left(\sigma Z_1-\left(\frac{\alpha}{\gamma}\right)^2 Z_1^2\right)\right] \\ &= \mathrm{sgn}\left[\xi\gamma_0\left(\sigma-\left(\frac{\alpha}{\gamma}\right)^2 Z_1\right)\right]=\mathrm{sgn}(\xi\gamma_0).\end{aligned} \tag{43}$$

Note that the last equality is a consequence of (41).

So it is possible to formulate

Theorem 2. If function $\psi(Z)$ intersects transversally the OZ axis in some point $Z_1>0$, there exists an open interval $I\subset R^1$ such that for $\gamma\in I$ the system (38) possesses a one-parameter family of stable quasiperiodic solutions.

Remark. It is obvious that as long as the parameters η and χ in formula (42) are small, a stable LC will occur for $\gamma<0$, but generally speaking this is not true.

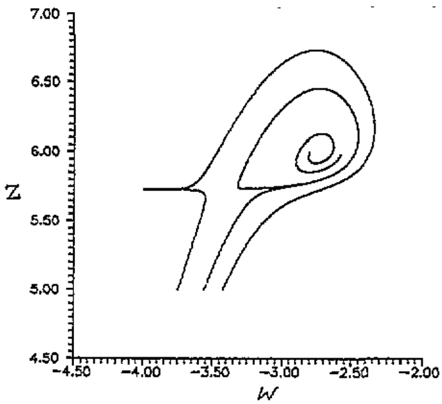

Figure 3. Phase portrait of system (38) in the vicinity of the critical point A for $\gamma=1,1$, $\gamma_0=1$, $\sigma=2$ and Ψ given by formula (44).

Figure 3 shows the phase portrait of system (38) in the vicinity of the critical point A for $\Psi(Z)$ given by the following expression:

$$\Psi(Z)=\begin{cases}2\sin\left\{\dfrac{\pi}{2}(Z-Z_1)\right\}\exp[0.7(Z-Z_1)^2] & \text{when } |Z-Z_1|\leqslant 2\\ 0 & \text{when } |Z-Z_1|>2.\end{cases} \tag{44}$$

Thus we have shown that the system (1)–(4) possesses invariant quasiperiodic solutions for kinetic equations of a quite general form.

Perhaps the necessary condition for the existence of quasiperiodic invariant solutions of hydrodynamical equations could be the presence of mass force in the Euler equation.

This supposition is backed by the fact that qualitative analysis of the invariant solutions of hydrodynamical equations without external forces has been performed many times (for comprehensive survey see [8], see also [11]), and quasiperiodic solutions were never found.

In conclusion let us briefly discuss the system (21)–(23) in the case when $g(L)\neq$constant. It is not difficult to see that all the critical points of this system are degenerate.

If, for any $L_1>0$ and $Z_1>0$, the equation $\Psi(L, Z)=(\sigma-1)Z$ is satisfied, then in the vicinity of Z_1 the function Ψ could be represented by (28) with $\xi=q(\sigma-1)g(L_1)\dot{\varphi}(Z_1)$, etc. And when the inequality (31) holds the linearization matrix of the system (21)–(23) in the vicinity of the critical point $A(W_1, Z_1, L_1)$ has one zero and two purely imaginary eigenvalues. As was shown in [12] such a degeneracy might be removed by a two-parameter family of small perturbations and, depending on the values of the parameters, one can obtain various solutions such as limit cycles, double cycles and, finally, homoclinic loops. The presence of the closed loops in the phase portrait of system (21)–(23) enables us to conclude that deterministic chaotic preturbulent solutions as well as quasiperiodic ones are inherent to the initial system.

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